Tutorium f. BMC Math. 2/3 6.5.26

Variation der Konstanten & Wronski - Determinante

Beispiele

$$y' + \frac{y}{x} = \cos(x) \quad \text{mit} \quad y(\pi) = 1$$

$$\boxed{y' + a(x)\cdot y = g(x) \implies y(x) = \left(k + \int e^{A(x)} \cdot g(x)\, dx\right) \cdot e^{-A(x)}}$$

wobei $A(x) = \int a(x)\, dx$ $\qquad g(x) = \cos(x)$

$$\Rightarrow \underbrace{y' + \frac{1}{x}\cdot y}_{\text{alt. Methode}} = \cos(x) \implies a(x) = \frac{1}{x} \implies A(x) = \ln(x)$$

$$-A(x) = -\ln(x)$$

alt. Methode

$$y' + \frac{1}{x}\cdot y = 0$$

$$\Rightarrow y' = -\frac{1}{x}\cdot y \implies \frac{1}{y}\, dy = -\frac{1}{x}\, dx \quad \int \ldots$$

$$y(x) = \left(k + \int e^{\ln(x)} \cdot \cos(x)\,dx\right) \cdot e^{-\ln(x)} \qquad \frac{1}{e^{\ln(x)}} = \frac{1}{x}$$

$$= \left(k + \int \underbrace{x \cdot \cos(x)}_{\text{Tabelle}}\,dx\right) \cdot \frac{1}{x}$$

$$= \left(k + \left(x \cdot \sin(x) + \cos(x)\right)\right) \cdot \frac{1}{x} \Rightarrow \frac{k}{x} + \sin(x) + \frac{\cos(x)}{x} = y(x)$$

mit $y(\pi) = 1 \Rightarrow 1 = \frac{k}{\pi} + \underbrace{\cancel{\sin(\pi)}}_{=0} + \frac{\cos(\pi)}{\pi} \Rightarrow 1 = \frac{k}{\pi} - \frac{1}{\pi}$

$$\Rightarrow 1 + \frac{1}{\pi} = \frac{k}{\pi} \Rightarrow \underline{\underline{k = \pi + 1}} \Rightarrow \underline{\underline{y(x) = \frac{\pi+1}{x} + \sin(x) + \frac{\cos(x)}{x}}}$$

$x \cdot y' + y = x^2 + 1 \quad | :x \qquad y(3) = 0 \qquad$ AWP

$y' + \frac{1}{x} \cdot y = x + \frac{1}{x} \qquad a(x) = \frac{1}{x} \Rightarrow A(x) = \ln(x) \qquad g(x) = x + \frac{1}{x}$

$-A(x) = -\ln(x)$

$y = \left(k + \int \underbrace{e^{\ln(x)}}_{x} \cdot \left(x + \frac{1}{x}\right) dx\right) \cdot \frac{1}{x}$

$= \left(k + \int x^2 + 1 \, dx\right) \frac{1}{x} = \left(k + \frac{1}{3}x^3 + x\right) \cdot \frac{1}{x} =$

$= \frac{k}{x} + \frac{1}{3}x^2 + 1 \Rightarrow 0 = \frac{k}{3} + \frac{1}{3} \cdot 9 + 1 \Rightarrow k = -12$

Typ rechte Seite

$$y'' + ay' + by = g(x)$$

$$y'' + 2y' + y = 2 \cdot e^{-x}$$

$e^{\lambda x}$

$\Rightarrow$ aus homog. DGL

$$\Longrightarrow \quad \lambda^2 + 2\lambda + 1 = 0 \quad \lambda = -1 \ \hat{=}\ c$$

doppelte Lösung

$$y_p = A\,\cancel{x^2} \cdot e^{-x} + B \qquad y_p' = A \cdot \left(2x \cdot e^{-x} - x^2 \cdot e^{-x}\right)$$

$$y_p'' = 2 \cdot A \cdot e^{-x} - 4 \cdot A \cdot x\, e^{-x} + A \cdot \cancel{x^2} \cdot e^{-x}$$

durch Vergleich

$B = 0$!

$$2y_p' = 4 \cdot A \cdot x\, e^{-x} - 2 \cdot A \cdot \cancel{x^2} \cdot e^{-x}$$

$$\Rightarrow 2 \cdot A \cdot e^{-x} - \cancel{4 \cdot A \cdot x \cdot e^{-x}} + \cancel{4 \cdot A \cdot x \cdot e^{-x}} + B = 2 \cdot e^{-x} + 0$$

$$\Rightarrow 2 \cdot A \cdot e^{-x} = 2 \cdot e^{-x} \qquad \Longrightarrow \underline{\underline{A = 1}}$$

$$y_p = x^2 \cdot e^{-x}$$

Wronski Det.

$$y'' - 4y' + 4y = e^{3x}$$

$\Rightarrow$ homog. Lösung: $\Rightarrow \lambda^2 - 4\lambda + 4 = 0 \Rightarrow \lambda_{1/2} = \underline{\underline{2}}$

$$\Rightarrow y_h = (c_1 + c_2 \cdot x) \cdot e^{2x} = c_1 \cdot \underbrace{e^{2x}}_{} + c_2 \cdot \underbrace{x \cdot e^{2x}}_{}$$

$$\det W = \begin{vmatrix} f_1 & f_2 \\ f_1' & f_2' \end{vmatrix} = \begin{array}{cc} f_1 & f_2 \\ \left| \begin{matrix} e^{2x} & x \cdot e^{2x} \\ 2e^{2x} & e^{2x} + 2x \cdot e^{2x} \end{matrix} \right| & \end{array} = e^{2x} \cdot \left(e^{2x} + 2x \cdot e^{2x}\right) - 2x\, e^{2x} \cdot e^{2x}$$

$$= e^{4x} + 2\cancel{x} \cdot e^{4x} - 2\cancel{x} \cdot e^{4x} = e^{4x} = W(x)$$

$$c_1 = -\int \frac{e^{3x} \cdot x \cdot e^{2x}}{e^{4x}}\,dx = -\int \frac{e^{5x} \cdot x}{\cancel{e^{4x}}}\,dx = -\int \underbrace{x \cdot e^{x}\,dx}_{\text{Tabelle}}$$

$$c_1 = -(x-1) \cdot e^{x} = \underline{\underline{(1-x) \cdot e^{x} = c_1}}$$

$$c_2 = \int \frac{e^{3x} \cdot e^{2x}}{e^{4x}}\,dx = \int e^{x}\,dx = \underline{\underline{e^{x} = c_2}}$$

$$\Rightarrow y_p = (1-x) \cdot e^{x} \cdot e^{2x} + e^{x} \cdot x \cdot e^{2x} = e^{3x} \cancel{- x \cdot e^{3x}} \cancel{+ x \cdot e^{3x}}$$

Probe: $\boxed{\underline{\underline{y_p = e^{3x}}}}$ $\quad y_p' = 3 \cdot e^{3x} \quad y_p'' = 9 \cdot e^{3x}$

$$\Rightarrow 9e^{3x} - \underbrace{4 \cdot 3}_{12} \cdot e^{3x} + 4 \cdot e^{3x} = 13 \cdot e^{3x} - 12 \cdot e^{3x} = \underline{\underline{e^{3x}}}$$

$$y'' + 9y = 3 \Rightarrow y_h \Rightarrow \lambda^2 + 9 = 0 \Rightarrow \lambda^2 = -9 \qquad \boxed{i^2 = -1}$$

$$\Rightarrow \underline{\underline{\lambda = \pm 3i}}$$

$$y_h = K_1 \cdot \overset{f_1}{\sin(3x)} + K_2 \cdot \overset{f_2}{\cos(3x)}$$

$$W(x) = \begin{pmatrix} \sin(3x) & \cos(3x) \\ 3 \cdot \cos(3x) & -3 \cdot \sin(3x) \end{pmatrix} = \underline{\underline{-3}} \qquad \left(\sin^2 + \cos^2 = 1\right)$$

$$K_1 = \frac{1}{3} \sin(3x)$$

$$K_2 = \frac{1}{3} \cos(3x) \qquad \Rightarrow \underline{\underline{y_p = \frac{1}{3}}}$$